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MAXIMALLY FAST BRAKING OF AN OBJECT IN CONTROLLED MOTION, UNDER THE ACTION OF AERODYNAMIC DRAG AND GRAVITY*

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In order to study controlled motion of objects in the atmosphere, which travel under the influence of aerodynamic drag and gravity, a model problem is used to investigate the mechanism by which these forces affect the intensity of braking of the object in an exponential atmosphere. A time-optimal control is synthesized for objects whose aerodynamic drag may be characterized exclusively as a force proportional to the product of the velocity of motion times the density of the atmosphere at the current altitude of motion, on the assumption that the atmosphere is exponential. The control synthesis, represented in generalized coordinates V, ρ, is independent of the braking characteristic T of the object and the parameter k characterizing the variation of atmospheric density; it is determined solely by the magnitude of the generalized terminal stopping velocity V_k and the values of $\rho_{\min/\max}$, which are determined by the position of the boundaries of the phase constraint (PC). It is shown by a numerical experiment how one can simplify the optimal synthesis by introducing a certain control "significance" level.

1. Statement of the problem. In connection with the control of motion in an exponential atmosphere (i.e. /1/, the density of the atmosphere varies exponentially with altitude H: $r = r_0 \exp(-kH)$), in a plane-parallel gravitational field, we shall consider the mechanism by which aerodynamic drag and the force of gravity affect the braking of an object. The model problem studied below may be given the following physical interpretation.

In an inertial coordinate system (see Fig.1, in which XOZ is the plane of the local horizon and the H axis points along the gravitational lines of force), an object P (the pursuer) moving at velocity V_p is approaching an object Z (the pursued object) which is moving at a velocity V_z. Initially, the object P receives a starting impulse which imparts to it an initial velocity V_{p0}. As it continues to move, its mass remains constant, but the direction of its velocity V_p is modified by its control system, which ensures that the projections of V_p and V_z on a plane perpendicular to the line joining P and Z (called the range line, CPZ) remain equal to all times. The magnitude of V_p decreases under the action of aerodynamic drag, whose magnitude may be considered proportional to the product $r_p V_p$

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(where r_p is the density of the atmosphere at the moving altitude of P), and the projection of the force of gravity on the direction of the velocity vector.

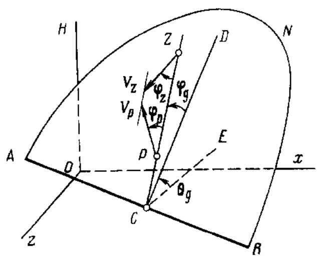

Fig.1

The object Z is moving at a velocity V_z of constant magnitude. The task of its control system is, by changing the direction of V_z (in an admissible range of angles φ_z, θ_g, see Fig.1), to decelerate P as rapidly as possible to the velocity $V_p = V_z$.

The control law exerted by P on Z is known.

Letting V_z have an arbitrary direction in inertial space and denoting the various angles as in Fig.1, we can determine the orientation (the angle φ_p in the plane N) of V_p relative to the range line (CPZ) and relative to the plane of the horizon (the angle θ_p, not shown in Fig.1). The angles are:

$$\varphi_p = \arcsin \frac{V_z \sin \varphi_z}{V_p}, \quad \sin \theta_p = \cos(\varphi_p + \varphi_g) \sin \theta_g$$

where the angle between the vector V_z in the plane N and the range line is φ_z; the plane N, which is determined by the range line and the vector V_z, makes an angle θ_g with the plane of the horizon (the line AB is the intersection of these planes; the line CD on N and the line CE in the plane of the horizon are perpendicular to AB, and the angle between the range line and CD is φ_g). It follows from the above arguments that the vector V_p must lie in N; as to the object P, the magnitude of its velocity V_p and its altitude H_p may be described by the following differential equations /1/:

$$V_p^{\bullet} = -Tr_pV_p - g\cos(\varphi_p + \varphi_g)\sin\theta_g$$
$$H_p^{\bullet} = V_p\cos(\varphi_p + \varphi_g)\sin\theta_g$$
$$(\ )^{\bullet} = d(\ )/dt, \quad T = \text{const}$$

and PC $H_{\min} \leqslant H_p \leqslant H_{\max}$, where $H_{\min}$ and $H_{\max}$ are the minimum and maximum admissible altitudes of motion of P, and g is the acceleration due to gravity.

Instead of the last equation and the PC, we can consider equations for the density r_p:

$$r_p^{\bullet} = -kr_pV_p\cos(\varphi_p + \varphi_g)\sin\theta_g$$
$$r_{\min} \leqslant r_p \leqslant r_{\max}, \quad r_{\min/\max} = r_0\exp(-kH_{\max/\min})$$

To investigate the braking mechanism, it will suffice to define the controlled quantity as $\Delta = \cos(\varphi_p + \varphi_g)\sin\theta_g$, which may vary within a fixed range $[\Delta_{\min}, \Delta_{\max}]$ containing zero. Defining

$$u = \Delta/u_0, \quad u_0 = \max(|\Delta_{\min}|, \Delta_{\max}) \tag{1.1}$$
$$u_{\min/\max} = \Delta_{\min/\max}/u_0; \quad V = ku_0V_p, \quad V_k = ku_0V_z; \quad \rho = Tr_p; \quad P_2 = gku_0^2$$

we obtain the following model optimal control problem for our investigation of the braking mechanism:

$$\min_u t_k \tag{1.2}$$
$$V^{\bullet} = -\rho V - P_2u, \quad V(t_k) = V_k; \quad \rho^{\bullet} = -\rho Vu$$
$$\rho_{\min} \leqslant \rho \leqslant \rho_{\max}; \quad u_{\min} \leqslant u \leqslant u_{\max} \tag{1.3}$$

This problem meets the conditions under which the method described in /2/ can be used for control synthesis. It differs from the problems discussed in /2/ in that the right-hand sides of Eqs.(1.2) involve non-linear (bilinear) terms. We shall investigate it for the case $\rho V > P_2$. Using the well-known formulae of the maximum principle /3/, we obtain:

adjoint differential equations

$$\psi_1^{\bullet} = (\psi_1 + \psi_2u)\rho, \quad \psi_2^{\bullet} = (\psi_1 + \psi_2u)V + \mu_1^{\bullet} - \mu_2^{\bullet} \tag{1.4}$$

where

$$\mu_1^{\bullet} \geqslant 0, \quad \mu_1^{\bullet}(\rho_{\min} - \rho) = 0, \quad \mu_2^{\bullet} \geqslant 0, \quad \mu_2^{\bullet}(\rho - \rho_{\max}) = 0 \tag{1.5}$$

and an optimal control

$$u^{\circ}(t) = \begin{cases} u_{\max}, \Pi(t) = (-\psi_1P_2 - \rho V\psi_2) > 0 \\ u_{\min}, \Pi(t) < 0 \end{cases} \tag{1.6}$$

If $t \geqslant t_n$, where t lies in a time interval in which $u^\circ(t) = u = \mathrm{const}$, it follows immediately from (1.4) that

$$\psi_1 + \psi_2 u = (\psi_1 + \psi_2 u)_{t=t_n} E(t, t_n) \tag{1.7}$$

$$\psi_1(t) = \psi_1(t_n) + (\psi_1 + \psi_2 u)_{t=t_n} \int_{t_n}^{t} \rho(\xi) E(\xi, t_n)\, d\xi \tag{1.8}$$

$$\psi_2(t) = \psi_2(t_n) + (\psi_1 + \psi_2 u)_{t=t_n} \int_{t_n}^{t} V(\xi) E(\xi, t_n)\, d\xi + \int_{t_n}^{t} (\mu_1^{\cdot} - \mu_2^{\cdot})\, d\tau \tag{1.9}$$

$$E(t, t_n) = \exp \int_{t_n}^{t} \{(\rho + uV) + (\mu_1^{\cdot} - \mu_2^{\cdot}) u\}\, d\tau$$

2. *Optimal control for motion within the PC.* It follows from the conditions of the maximum principle that the optimal control $u^\circ(t)$ will be a step function.

Proposition. The optimal control $u^\circ(t)$ has at most one switching.

Proof. It follows directly from (1.2), (1.4) that

$$\Pi^{\cdot} = -\psi_1 (P_2\rho + \rho V^2) + \psi_2 \rho^2 V \tag{2.1}$$

if $\Pi^{\cdot} = 0$ then $\Pi(t) = -\psi_1 (2P_2 + V^2)$ (2.2)

if $\Pi = 0$ then $\Pi^{\cdot} = [-\psi_1 (2P_2 + V^2)]\, \rho$ (2.3)

and if $\psi_1(t^*) = \psi_2(t^*) = 0$, then it follows from (1.8) and (1.9) that $\psi_1(t) \equiv \psi_2(t) \equiv 0,\ t \geqslant t^*$.

We assert that the function $\Pi(t)$ has no further zeros for $t > t_n$ $(\Pi(t_n) = 0)$. Then necessarily

$$\psi_1(t_n)\psi_2(t_n) < 0, \quad |\psi_1(t_n)| > |\psi_2(t_n)|$$

Suppose that at a time t_n^+, i.e., immediately after the switching, $u^\circ = u_{\max}$.

Only the following cases are possible:

$$01^+\colon \psi_1(t_n) > 0, \quad \psi_2(t_n) < 0, \quad \psi_1(t_n^+) + \psi_2(t_n^+)\, u_{\max} > 0$$

$$02^+\colon \psi_1(t_n) < 0, \quad \psi_2(t_n) > 0, \quad \psi_1(t_n^+) + \psi_2(t_n^+)\, u_{\max} < 0$$

where the last inequalities follow from the conditions $\Pi(t_n) = 0$ and $\rho V > P_2 > 0$.

We shall use the relations (1.7)-(1.9).

Case 01^+. The functions ψ_1, ψ_2 are monotone increasing. By condition (2.1), before $\psi_2(t)$ changes sign, necessarily $\Pi^{\cdot} < 0$, and consequently when $t > t_n$ then $\Pi(t)$ cannot be positive. Consequently, under condition 01^+ the control cannot switch to $u^\circ = u_{\max}$.

Case 02^+. The functions ψ_1, ψ_2 are monotone decreasing. By (2.1), before $\psi_2(t)$ changes sign, necessarily $\Pi^{\cdot} > 0$, i.e., $\Pi(t) > 0$, and when $t > t_n$ it is monotone increasing. If $\Pi(t)$ experiences a new change of sign, then at the corresponding point $t = t^{**}$ necessarily $\Pi^{\cdot}(t^{**}) < 0$. But it follows from (2.3) and the condition $\psi_1(t^{**}) < 0$ that $\Pi(t^{**}) > 0$. This contradiction proves that the function $\Pi(t)$ has no new zero. Thus, after the switching to $u^\circ = u_{\max}$ no further switching will occur.

Suppose that at time $t = t_n^+$, i.e., immediately after the switching, $u^\circ = u_{\min}$. The only possible cases are 01^- and 02^-, which are obtained from cases 01^+ and 02^+ by replacing $u_{\max}$ with $u_{\min}$.

Case 01^-. The functions $\psi_1(t), \psi_2(t)$ are monotone increasing. By (2.1), before $\psi_2(t)$ changes sign, necessarily $\Pi^{\cdot} < 0$ and, consequently, $\Pi(t)$ cannot change sign. It is negative in this interval, and after $\psi_2(t)$ changes sign it cannot change sign again. Hence there is no switching.

Case 02^-. The functions $\psi_1(t), \psi_2(t)$ are monotone decreasing. By (2.1), before $\psi_2(t)$ changes sign, necessarily $\Pi^{\cdot} > 0$. Consequently, after vanishing at $t = t_n$ the function $\Pi(t)$ must increase and cannot be negative again, i.e., under the conditions of case 02^- no switching to $u^\circ = u_{\min}$ is possible.

We have thus examined all possible cases of the optimal motion after switching has occurred. We have established that after the switching no second switching (change in the sign of $\Pi(t)$) can occur. It follows that in the optimal motion there cannot be more than one switching within the PC.

It follows from this proposition that the function $\Pi(t)$, after reaching zero, will never vanish again.

3. *Optimal control for motion along the PC.* Let us assume that the system, in the course of its optimal motion, reaches the left $\rho = \rho_{\min}$ of the PC. Arrival at the PC occurs when $u^\circ(t) = u_{\max}$, motion along the PC may take place (by (1.2)) only if $u = 0$ and if $\mu_1^\cdot \geqslant 0$, $\mu_2^\cdot = 0$. At the time t_f of arrival at the PC it must be true that

$$\Pi(t_f^-) \geqslant 0, \quad \Pi(t_f^+) = 0; \quad \psi_1(t_f^-) = \psi_1(t_f^+)$$
$$\psi_2(t_f^+) = \psi_2(t_f^-) + \int_{t_f^-}^{t_f^+} \mu_1^\cdot d\tau$$

and since

$$\Pi(t_f^+) = \Pi(t_f^-) - \rho V \int_{t_f^-}^{t_f^+} \mu_1^\cdot d\tau$$

it follows that owing to the "jump" of $\mu_1^\cdot$ one can guarantee that $\Pi(t_f^+) = 0$. The identity $\Pi(t) \equiv 0$ will hold in the interval $t_f \leqslant t \leqslant t_s$, where t_s is the time the system leaves the PC, provided that

$$\psi_1(t) = \psi_1(t_f) E_{\min}$$
$$\psi_2(t) = \psi_2(t_f^-) + \psi_1(t_f) V(t_f)(t - t_f) + \int_{t_f^-}^{t} \mu_1^\cdot d\tau$$
$$E_{\min} = \exp\{\rho_{\min}(t - t_f)\}$$

which gives

$$\Pi(t) = -\psi_1(t_f)\{P_2 E_{\min} + V(t_f)\rho_{\min} V(t - t_f)\} - \psi_2(t_f^-)\rho_{\min} V - \rho_{\min} V \int_{t_f^-}^{t} \mu_1^\cdot d\tau \tag{3.1}$$

Let us consider the possibility of ensuring that $\Pi(t) \equiv 0$ in (3.1) by varying $\mu_1^\cdot \geqslant 0$. To that end the following equalities, which follow from (1.6), (3.1), must hold:

$$\Pi(t) \equiv 0 = -\psi_1 P_2 - \rho V \psi_2, \quad \Pi^\cdot(t) \equiv 0 = -\psi_1 (P_2 \rho_{\min} + \rho_{\min} V^2) + \psi_2 \rho_{\min}^2 V - \rho_{\min} V \mu_1^\cdot \tag{3.2}$$

Hence

$$-\psi_1(2P_2 + V^2) - V\mu_1^\cdot = 0 \tag{3.3}$$

It follows from this equality that at the time t_f^+ of arrival at the PC necessarily $\Pi(t_f^+) = 0$, and the condition $\psi_1(t) \leqslant 0$ must hold in $t_f < t < t_s$; this does not contradict the optimum conditions listed above.

For departure from the PC necessarily $\Pi(t_s^+) < 0$, which may be achieved thanks to the "jump" of $\mu_1^\cdot \geqslant 0$.

Note that because $\Pi(t) = 0$ along the PC, no further switching can occur after departure from the PC.

Suppose that the system, moving in an optimum fashion, reaches the right $\rho = \rho_{\max}$ ($\mu_1^\cdot = 0$, $\mu_2^\cdot \geqslant 0$). This may occur with $u^\circ(t) = u_{\min}$, i.e., $\Pi(t_f^-) = -\psi_1 P_2 - \rho V \psi_2 \leqslant 0$. It is obvious from the adjoint differential equations that the necessary condition for motion along $\rho = \rho_{\max}$, i.e., $\Pi(t) \equiv 0$, may be guaranteed by making

$$\psi_1(t) = \psi_1(t_f) E_{\max}, \quad E_{\max} = \exp\{\rho_{\max}(t - t_f)\}$$
$$\psi_2(t) = \psi_2(t_f^-) + \psi_1(t_f) V(t - t_f) - \int_{t_f^+}^{t} \mu_2^\cdot d\tau, \quad \mu_2^\cdot \geqslant 0$$

The equalities obtained from (3.2) by replacing $\rho_{\min}$ with $\rho_{\max}$ and $\mu_1^\cdot$ with $(-\mu_2^\cdot)$ must hold on the PC. These equalities yield an expression analogous to (3.3), with $\mu_1^\cdot$ replaced by $(-\mu_2^{\cdot})$.

This requires that at the time t_f^+ of arrival at the PC necessarily $\Pi(t_f^+)=0$, and in the interval $[t_f^+, t_s^-]$ we must have $\psi_1(t)\geqslant 0$, which does not contradict the optimum conditions listed above.

For departure from the PC necessarily $\Pi(t_s^+)\geqslant 0$, which may always be achieved by a "jump" of $\mu_2^{\cdot}\geqslant 0$. Note that since $\Pi(t)=0$ on the PC, no further switching is possible after departure from the PC.

*4. **The structure of the optimal control.*** It has been established by an analysis of the optimality conditions that: the optimal control is a step function and it can have at most one switching; if the system hits the PC one must have $u^\circ=0$ and with this control the system may halt at the constraint before the end of the control process; departure from the PC is possible, but there can be no further switchings within the PC.

To estimate the sign of the optimal control with which the system starts its optimal motion, it is convenient to employ the following arguments: there can be at most one switching; over a finite part of the motion, from $V_k+\Delta V$ to V_k, provided ΔV is sufficiently small, the motion must proceed with a u° value which minimizes the duration of the motion:

$$\min_{u_{\min}\leqslant u\leqslant u_{\max}} \frac{\Delta V}{\rho V+P_2 u}, \quad \text{i.e.} \quad u^\circ=u_{\max}$$

These arguments imply that if the optimal motion occurs with one switching (which is unique!), then it must begin with $u^\circ=u_{\min}$. There must exist a switching curve (SC) in the V,ρ plane, above which (with respect to the coordinate V) $u^\circ=u_{\min}$, the motion here may hit only the right PC, moving along it up to the SC and leaving it immediately after the phase point crosses the SC (this has been confirmed by numerical experiment: the duration of motion along the PC exceeds the duration of motion with $u^\circ=u_{\max}$). Below the SC the motion may hit only the left PC, subsequently moving along it until the control process ends.

*5. **Investigating the properties of the optimal control.*** With an eye to the questions that arise in different physical interpretations of the model problem, we will present the results of a study bearing on the following questions: just how "significant" is the occurrence of a switching in the optimal control? If it is significant, how accurately must one then determine the SC? In what plane N (Fig.1) is the manoeuvring of the object Z most effective with respect to the criterion being studied?

Our study of the SC and the properties of the optimal control will apply to the range $P_2=0.0004\ldots0.0010$, $V_K=0.01\ldots0.08$, $|u_{\max/\min}|=1$.

5.1. Approximate description of the SC for large ρ and large V. In (V,ρ) coordinates, the SC has the form of a monotone decreasing concave curve (see the upper part of Fig.2, where the SC is shown for $P_2=0.001$ and a few values of V_k; the figures above the curves indicate values of $10^2 V_k$) with horizontal and vertical asymptotes. A change in V_k, while preserving the general shape of the SC, alters its position in the (V,ρ) plane. As V_k is increased the SC is "pulled up" along the V axis.

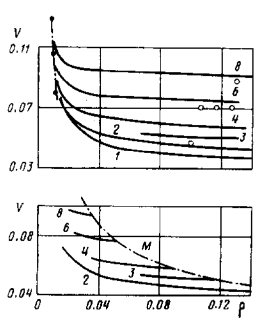

Fig.2

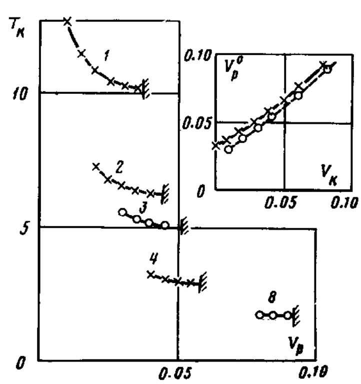

Fig.3

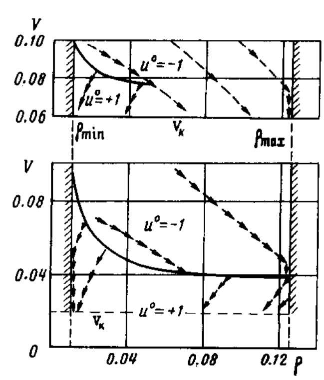

Fig.4

SC at $\rho\geqslant 0$. Bellman's equation for the minimum travel time ω of the phase point from the SC to $V=V_k$, which occurs with $u^\circ=+1$, is

$$\frac{\partial\omega}{\partial V}(-\rho V-P_2)+\frac{\partial\omega}{\partial\rho}(-\rho V)=1, \quad \omega(V_k,\rho)=0 \tag{5.1}$$

and the following equality must hold on the SC:

$$\frac{\partial\omega}{\partial V} P_2 + \frac{\partial\omega}{\partial\rho} \rho V = 0 \tag{5.2}$$

The equations of the characteristics of (5.1) and the assumption that near the SC $V - \rho = C_1$ imply

$$V - \rho = C_1, \quad \omega(V_k, \rho) = 0, \quad C_1 = \text{const}$$
$$\frac{1}{C_1} \ln \frac{\rho}{\rho + C_1} = -\omega + C_2$$

Eliminating the constants of integration C_1, C_2 and using Eq.(5.2), we obtain the following approximate equation of the SC at $\rho \gg 0$:

$$\frac{\rho V - P_2}{V - \rho} \ln\left[\frac{V}{\rho} \frac{V_k - (V - \rho)}{V_k}\right] + \left(\frac{P_2}{V} - V\right) + \frac{\rho V - P_2}{V_k - (V - \rho)} = 0 \tag{5.3}$$

The open circles in the upper part of Fig.2 show the results of a control comparison of the points of the approximate SC (5.3) with those of the true SC. The calculations, done with $V_k = 0.08, 0.06, 0.03$, justify the use of (5.3) when $\rho \geqslant 0.1$.

The SC for $V \gg 0$. As shown by numerical experiment, this part of the SC is independent of V_k and is satisfactorily described by the appropriate part of the SC for $V_k = 0.01$ or the function $\rho V = 10^{-3}$. It is shown in the upper part of Fig.2 (toward the left) by the dash-dotted curve through the closed circles.

5.2. Estimation of the "significance" of the SC (the case without a PC). The desirability of switching in general will be measured by the following criterion: the difference in time between the duration TW of the control process without switching and the minimum possible time T°, i.e., the duration of the process with optimum switching time.

Let us stipulate that the "significant" value of the difference is $(TW - T^\circ) = 1$ sec. Then an optimal control will be considered significant only if $(TW - T^\circ) \geqslant 1$ sec. A synthesis of this control for $P_2 = 0.001$ is shown in the lower part of Fig.2. For each V_k value (shown by specifying $10^2 V_k$ alongside each SC) the figure shows only those parts of the SC along which one can obtain $TW - T^\circ \geqslant 1$ sec. Outside this part of the SC, switching does not yield the expected savings in time. The dotted curve M in the lower part of Fig.2 (numerically calculated) represents the locus of the right endpoints of the "active" SCs.

5.3. Approximate SC in the presence of a PC. The next question we consider is of importance if we wish to simplify the synthesis: assuming that the right PC is reached, can one simplify the rule for leaving it? To that end, let us compare the time required for the phase point, having hit the PC, to move along it until the control process ends, with the time required for it to move first along the PC and then, after leaving the latter, along a trajectory within the PC with $u^\circ = +1$.

A numerical experiment was used to estimate the time T_k for the system to move from the point at which the SC intersects the right PC $\rho = \rho_{max}$ (the V coordinate of this point is V_p°) to the end of the control process (sectors of the motion: along the PC, departure from it at $V = V_p$ and further motion along a path with $u^\circ = +1$), as a function of the point V_p of departure from the PC. It is obvious from Fig.3 (the figure above each curve in the T_k, V_p plane represents the appropriate value of $10^2 V_k$) that as the point of departure moves from $V_p = V_k$ to $V_p = V_p^\circ$ (the hatched ends of the curves) the time of motion T_k decreases monotonically; the rate of decrease increases as V_k decreases. At the same time one sees that if the optimum time can be determined to within 1 sec, then the "accuracy" with which one can maintain the ordinate of the point of departure from the PC may be $V_p^\circ - V_p \approx 0.02$. Accuracy of this order is indeed provided by the above formula for the sector of the SC with $\rho \gg 0$ (see the upper right part of Fig.3: the solid curve is the exact value of V_p° as a function of V_k and the dashed curve is the approximate curve obtained by formula (5.3); the results are shown for $\rho_{max} = 0.125$).

If one is satisfied with a control capable of guaranteeing a travel time differing from optimum by at most 1 sec, then departure from the PC is justifiable only when $V_k \leqslant 0.02$. Otherwise the system can move after V_p° without leaving the PC. Thus, synthesis of an optimal control with significance level $TW - T^\circ = 1$ sec will look something like the situation in Fig.4. The dashed curves represent typical sections of optimal trajectories; the arrows represent the direction of motion of the phase point along them, the labels -1 and +1 mark regions in which $u^\circ = -1$ and $u^\circ = +1$, respectively, the upper part of the figure corresponds to $V_k > 0.02$ and the lower part to $V_k \leqslant 0.02$.

5.4. Optimal "manoeuvring plane" (Fig.1). The effect of the inclination of the plane N (the change in the angle θ_g is measured by the parameter P_2) on the duration of the control process was verified at a point: the starting initial point $V_{n0} = 0.10$ and the starting terminal point $V_{k0} = 0.02$.

For a numerical comparison of the length T° of the control process one has to take into account conditions (1.1), from which it follows that, in order to keep the problem physically meaningful, the variation of P_2 from its starting value P_{20} to P_2 must be accompanied by changes in the starting value V_n and terminal value V_k of the velocity $V_n = V_{n0}\sqrt{P_2/P_{20}}$; $V_k = V_{k0}\sqrt{P_2/P_{20}}$. The results of the computations for $P_{20} = 0.001$, $V_{n0} = 0.1000$, $V_{k0} = 0.0200$, $\rho_0 = 0.013$ were as follows:

$P_2 \cdot 10^4$	10	8	6	4
$V_n \cdot 10^2$	10.00	8.94	7.75	6.33
$V_k \cdot 10^2$	2.000	1.788	1.550	1.266
V_n/V_k	5	4.994	5	5
T°	31.89	34.29	37.56	42.35

The row V_n/V_k in the table serves as a control. The computations showed that the duration T° of the process increased monotonically as P_2 was reduced. Thus, to achieve maximally fast braking of the object P, the most preferable situation for Z is a vertical manoeuvring plane (the plane N with maximum P_2).

6. Conclusion. Considering a class of objects moving under the influence of an undesirable resistance force (proportional) to the velocity of the object and the density of the atmosphere at its altitude at any given time) and the force of gravity, we have been able to propose the synthesis of an optimal control ("optimal" in the sense of maximally fast braking; the control is implemented by adjusting the spatial orientation of the object's velocity vector) in generalized coordinates $V = ku_0V_l$, $\rho = Tr_p$ and with the parameter $P_2 = gku_0^2$.

The optimal control has at most one switching point within the phase constraints (there exists a switching curve (SC) in the V, ρ plane) and it steers the object at most once to the phase constraints.

The position of the SC in the generalized coordinate plane V, ρ depends only on the generalized terminal velocity V_k down to which the object is to be decelerated. For large V, ρ values, a good description of the SC is given by the analytical expressions obtained above.

In the practical use of this synthesis technique, attention should be paid to the fact that for every fixed value of V_k and P_2 there is a $\{V, \rho\}$ region in which one can drop a) the requirement that the control be switched at some time, or b) the requirement that the system leave the boundary of the right phase constraint, without significantly affecting the time elapsing until the completion of the control process (i.e., only part of the SC need actually be realized).

With regard to the "physical problem" described at the beginning of the paper, we have shown that the best position of the manoeuvring plane from the standpoint of the object Z is the vertical plane.

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